

SEIFERT MATRICES AND 6-KNOTS

J. A. HILLMAN AND C. KEARTON

ABSTRACT. A new classification of simple \mathbf{Z} -torsion-free $2q$ -knots, $q \geq 3$, is given in terms of Seifert matrices modulo an equivalence relation. As a result the classification of such $2q$ -knots, $q \geq 4$, in terms of F -forms is extended to the case $q = 3$.

0. Introduction. An n -knot k is an oriented locally flat PL sphere-pair (S^{n+2}, S^n) ; equally, one could consider oriented smooth pairs where the embedded sphere is allowed to carry an exotic differentiable structure. Let K denote the closed complement of a regular neighbourhood of S^n , often called the *exterior* of k . Then k is *simple* if K has the homotopy $[(n-1)/2]$ -type of a circle; this is the most that can be asked without making k unknotted. With the exception of $n = 4$ and 6, the simple n -knots ($n \geq 3$) have been classified in various ways during the past twenty years. The first such result is due to J. Levine [L2], who classified the simple $(2q-1)$ -knots ($q \geq 2$) in terms of the Seifert matrix and S -equivalence. These knots were then classified in terms of the Blanchfield duality pairing in [T1, T2, and K1]. Results for certain classes of simple $2q$ -knots ($q \geq 4$) may be found in [K2, Ko1, Ko2, and K3]; the general case is given by M. Sh. Farber in [F2]. One should also mention here the pioneering work of M. A. Kervaire [Ke], who characterised the homology modules which can occur, in terms of presentation matrices.

Let $\tilde{K} \rightarrow K$ be the infinite cyclic (= universal) cover of K , where k is a simple $2q$ -knot, $q \geq 3$. Then k is \mathbf{Z} -torsion-free if $H_q(\tilde{K})$ has no \mathbf{Z} -torsion. In this paper we classify such knots in terms of Seifert matrices and an equivalence relation which we call F -equivalence. These matrices yield a presentation of the F -form of k , which is used in [K3] to classify these knots for $q \geq 4$. It is essentially the same as the Λ -quintet which Farber uses in [F2], although his result is not restricted to the \mathbf{Z} -torsion-free case. By the geometric results for $q \geq 4$, there is a one-one correspondence between F -equivalence classes of Seifert matrices and F -forms, and so the results of [K3] also hold for $q = 3$.

1. The Seifert linking form. Let k be a simple \mathbf{Z} -torsion-free $2q$ -knot, $q \geq 3$; then by Theorem 2 and Lemma 5 of [L2] there exists a Seifert surface V of k which is $(q-1)$ -connected and for which $H_q(V) \cong \pi_q(V)$ has no \mathbf{Z} -torsion.

Let $u \in H_q(V)$, $v \in H_{q+1}(V)$, and let $i_+ : H_*(V) \rightarrow H_*(S^{2q+2} - V)$ denote the map induced by "pushing off" in the positive normal direction. Then u and $i_+(v)$ are represented by disjoint cycles in S^{2q+2} , hence have a linking number taking

Received by the editors August 17, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57Q45.

This paper was conceived while the first author was an S.E.R.C. Visiting Fellow at Durham University.

values in \mathbf{Z} . Thus we obtain a linking pairing $L: H_{q+1}(V) \times H_q(V) \rightarrow \mathbf{Z}$ given by $L(v, u) = \text{link}(i_+(v), u)$.

Similarly, let $i_+: \pi_*(V) \rightarrow \pi_*(S^{2q+2} - V)$ denote the map obtained by “pushing off” in the positive normal direction. (The notation will cause no difficulty.) Any two elements $\mu, \nu \in \pi_{q+1}(V)$ can be represented by embedded spheres, and any S^{q+1} is unknotted in S^{2q+2} because the codimension is at least 3. Thus μ and $i_+(\nu)$ are represented by disjoint embeddings in S^{2q+2} , and $i_+(\nu)$ is an element of $\pi_{q+1}(S^{2q+2} - \text{Im } \mu) \cong \pi_{q+1}(S^q) \cong \mathbf{Z}/2\mathbf{Z}$. Thus we have a homotopy linking $\mathcal{L}: \pi_{q+1}(V) \times \pi_{q+1}(V) \rightarrow \mathbf{Z}/2\mathbf{Z}$ given by $(\nu, \mu) = \text{homotopy link}(i_+(\nu), \mu)$.

By a result of Whitehead [W, p. 555], there is a short exact sequence

$$\mathcal{E}: H_q(V)/2H_q(V) \xrightarrow{\sigma} \pi_{q+1}(V) \xrightarrow{\eta} H_{q+1}(V)$$

since $H_q(V) \cong \pi_q(V)$ by the Hurewicz theorem. Let $\tau: H_q(V) \rightarrow H_q(V)/2H_q(V)$ denote the quotient map. Then it is easy to see that

$$\mathcal{L}(\nu, \sigma\tau(u)) = L(\eta(\nu), u) \pmod{2}.$$

All this data is a *Seifert linking form*: $(H_q(V), H_{q+1}(V), L, \mathcal{L}, \mathcal{E}, \tau)$. Two such are *isomorphic* if there are isomorphisms of the groups which commute with all the appropriate maps and preserve L and \mathcal{L} .

PROPOSITION 1.1. *The Seifert linking form determines V up to ambient isotopy.*

PROOF. Suppose that k, k' are two such knots with $(q-1)$ -connected Seifert surfaces V, V' giving rise to isomorphic Seifert linking forms. Let $\varphi: H_*(V) \rightarrow H_*(V')$ be the isomorphism on homology, so that $L(v, u) = L'(\varphi(v), \varphi(u))$. Choose a basis u_1, \dots, u_n for $H_q(V)$ and set $u'_i = \varphi(u_i)$ for $1 \leq i \leq n$ to obtain a basis for $H_q(V')$. Now let B^{2q+1} be a closed $(2q+1)$ -ball in the interior of V ; then $H_q(V) \cong H_q(V, B^{2q+1})$ by the long exact sequence of homology, and so we obtain a basis $\tilde{u}_1, \dots, \tilde{u}_n$ for $H_q(V, B^{2q+1})$. By handlebody theory, there is a handle decomposition of V based on B^{2q+1} involving only handles of index q and $q+1$, say $h_1^q, \dots, h_n^q, h_n^{q+1}, \dots, h_n^{q+1}$, such that the core C_i of h_i^q represents \tilde{u}_i . Now $\partial C_1, \dots, \partial C_n$ are a set of disjoint $(q-1)$ -spheres embedded in $\partial B^{2q+1} \cong S^{2q}$; since $q \geq 3$, they are unlinked and isotopic to a standard set of such spheres. That is, they are isotopic to the boundaries of a set of n disjoint q -balls embedded in S^{2q} . Thus we can ambient isotop V' so that it coincides with V on the closed ball B^{2q+1} in its interior, and then (in the obvious notation for a corresponding handle decomposition of V') we can isotop $\partial C'_i$ to coincide with ∂C_i in ∂B^{2q+1} for $1 \leq i \leq n$. Now we would like to isotop the core C'_i to coincide with C_i ; standard isotopy theorems enable us to do this, for we can thicken up B^{2q+1} to $B^{2q+1} \times B^1 \subset S^{2q+2}$, and consider the C_i as embedded in $A = \text{cl}[S^{2q+1} - (B^{2q+1} \times B^1)]$. Then C'_1 can be homotoped to C_1 keeping the boundary fixed by Theorem 10.1 of [H]. If A_1 is A with an open neighbourhood of C_1 excised, then A_1 is q -connected, and so C'_2 can be homotoped to C_2 in A_1 keeping the boundary fixed, and hence isotoped to C_2 in A_1 keeping the boundary fixed. Continuing in this way, we can isotop $C'_1 \cup \dots \cup C'_n$ onto $C_1 \cup \dots \cup C_n$ keeping the boundary fixed. By the argument of Levine [L2, §16], the obstruction to isotoping the i th q -handle of V' onto h_i^q lies in $\pi_q(S^{q+1}) = 0$, and so we can do this for each i .

Now we denote the $(q+1)$ -handles of V (respectively V') by h_1, \dots, h_n (respectively h'_1, \dots, h'_n). Our object is to isotop h'_i onto h_i , but first we must select our handles carefully. The basis u_1, \dots, u_n of $H_q(V)$ yields a dual basis v_1, \dots, v_n of $H_{q+1}(V)$ under the intersection pairing $H_{q+1}(V) \times H_q(V) \rightarrow \mathbf{Z}$. Set $v'_i = \varphi(v_i)$, $1 \leq i \leq n$. Let $M = B^{2q+1} \cup h_1^q \cup \dots \cup h_n^q$, and similarly for M' ; note that M coincides with M' . Since (V, M) is q -connected, we have $H_{q+1}(V) \cong H_{q+1}(V, M) \cong \pi_{q+1}(V, M)$. We can choose h_1, \dots, h_n so that the core c_i of h_i realises the element $\tilde{v}_i \in \pi_{q+1}(V, M)$ corresponding to v_i under this isomorphism. And we make the corresponding choice for h'_i, \dots, h'_n , so that c'_i represents the element $\tilde{v}'_i \in \pi_{q+1}(V', M')$.

Because the q -handles are unknotted, we see that $M \cong \# \partial(S^q \times B^{q+1})_i$, the boundary connected sum taken over $i = 1$ to n , and so $\partial M \cong \#(S^q \times S^q)_i$. Clearly u_i , regarded as an element of $H_q(V)$ or $H_q(M)$ or $H_q(\partial M)$, is represented by $(S^q \times \text{point})_i$. The homology class of $d_i = (\text{point} \times S^q)_i$ being denoted by $w_i \in H_q(\partial M)$, and $H_q(\partial M)$ being identified with $\pi_q(\partial M)$ by the Hurewicz isomorphism, it follows at once that $\partial \tilde{v}_i \in \pi_q(\partial M)$ lies in the subgroup $\langle w_1, \dots, w_n \rangle$, for otherwise the attaching sphere of h_i would represent a nonzero element of $\pi_q(M) \cong H_q(M)$, and so $H_q(V)$ would not be a free abelian group of rank n . In fact, since v_1, \dots, v_n is dual to u_1, \dots, u_n , it follows that $\partial \tilde{v}_i = w_i$.

Thus the attaching sphere ∂c_1 of h_1 is homotopic to d_1 in ∂M . But $d_1 \cong S^q$ is $(q-1)$ -connected, ∂M is $(q-1)$ -connected, and $q \geq 3$, so we can apply Theorem 10.1 of [H] to show that ∂c_1 is ambient isotopic to d_1 in ∂M . The attaching sphere ∂c_2 of h_2 is homotopic to d_2 in ∂M ; we need to show that ∂c_2 is homotopic to d_2 in $\partial M - d_1$. Clearly $\pi_q(\partial M - d_1) \cong \pi_q(\partial M) / \langle u_1 \rangle \cong \langle u_2, \dots, u_n, w_1, \dots, w_n \rangle$, regarding u_1 as an element of $\pi_q(\partial M)$. Hence ∂c_2 is indeed homotopic to d_2 in $\partial M - d_1$. And hence ∂c_2 is isotopic to d_2 in $\partial M - d_1$, using Hudson's result again. Continuing in this way we can isotop $\partial c_1 \cup \dots \cup \partial c_n$ onto $d_1 \cup \dots \cup d_n$. The same applies to $\partial c'_1 \cup \dots \cup \partial c'_n$.

Next we wish to isotop h_i onto h'_i , keeping the attaching sphere fixed. Begin with h_1 : the difference between the homotopy class of $c_1 \text{ rel } \partial$ in $N_1 = \text{cl}[S^{2q+2} - M \times B^1]$ and that of c'_1 can be interpreted in terms of $L - L'$. Since the linking forms are isomorphic, the difference is zero, and so c_1 is homotopic rel ∂ to c'_1 , and by Hudson's result the homotopy can be realised by an ambient isotopy. Let N_2 be the closed complement of a regular neighbourhood of $M \cup h_1$ (rel ∂); then the difference in the homotopy classes of c_2 and c'_2 (rel ∂) in N_2 can be interpreted in terms of $L - L'$ and $\mathcal{L} - \mathcal{L}'$. Hence c_2 is homotopic, and so isotopic, to $c'_2 \text{ rel } \partial$ in N_2 . Continuing in this way we obtain an ambient isotopy taking each c_i onto c'_i . Finally we can isotop h_i onto h'_i as in §16 of [L2], the obstruction being $\mathcal{L}(\delta_i, \delta_i) - \mathcal{L}'(\delta'_i, \delta'_i) = 0 \in \pi_{q+1}(S^q)$, where $\delta_i \in \pi_{q+1}(V)$ is represented by $c_i \cup (\text{point} \times B^{q+1})_i \cong S^{q+1}$. Q.E.D.

2. Nice bases. Let V be a $(q-1)$ -connected Seifert surface of the \mathbf{Z} -torsion-free simple $2q$ -knot k , with $H_q(V)$ torsion-free. Consider $\mathbf{u} = \{u_1, \dots, u_n\} \subset H_q(V)$, $\mathbf{v} = \{v_1, \dots, v_n\} \subset H_{q+1}(V)$, $\boldsymbol{\nu} = \{\nu_1, \dots, \nu_n\} \subset \pi_{q+1}(V)$. We say that \mathbf{u} and \mathbf{v} are *nice bases*, and that $\boldsymbol{\nu}$ *lies over* \mathbf{v} if the following properties hold.

- (i) \mathbf{u} and \mathbf{v} are dual bases of $H_q(V)$ and $H_{q+1}(V)$;
- (ii) the u_i are represented by disjoint embedded spheres S_i^q ;
- (iii) the ν_i are represented by disjoint embedded spheres S_i^{q+1} ;

- (iv) S_i^q meets S_j^{q+1} in exactly δ_{ij} points;
- (v) ν_i is mapped onto v_i by the Hurewicz homomorphism.

PROPOSITION 2.1. *Let \mathbf{u} be a basis of $H_q(V)$. If $q \geq 3$, then there exist \mathbf{v} and ν such that \mathbf{u} and \mathbf{v} are nice bases, and ν lies over \mathbf{v} .*

This result is implicit in the proof of Proposition 1.1.

Let ς be the nontrivial element of $\pi_{q+1}(S^q)$, where $q \geq 3$. By the Hurewicz theorem, $H_q(V) \cong \pi_q(V)$, so thinking of u_i as an element of $\pi_q(V)$, we obtain an element $u_i \circ \varsigma \in \pi_{q+1}(V)$.

PROPOSITION 2.2. *Let $q \geq 3$, and suppose that \mathbf{u} and \mathbf{v} are nice bases with ν lying over \mathbf{v} . Then ν can be modified in either of the following two ways to obtain ν' , also lying over \mathbf{v} .*

- (i) Replace ν_i by $\nu'_i = \nu_i + u_i \circ \varsigma$.
- (ii) For $i \neq j$, replace ν_i by $\nu'_i = \nu_i + u_j \circ \varsigma$ and ν_j by $\nu'_j = \nu_j + u_i \circ \varsigma$.

PROOF. (i) The homotopy class ν'_i can be represented by an embedding in the complement of the other $S_j^{q+1} \cup S_j^q$, using Theorem 8.1 of [H]. Then $S_i^q \cap (\text{new } S_i^{q+1})$ can be reduced to one point by the Whitney trick.

(ii) Let $A = S_i^q \cap S_i^{q+1}$, $B = S_j^q \cap S_j^{q+1}$, and choose a path from A to B which misses the spheres apart from its endpoints. Take a regular neighbourhood N of this path, meeting the spheres regularly. Regard N as $B^{2q} \times I$, so that $\partial N \cap (S_j^q \cup S_j^{q+1})$ is a pair of once linked spheres in $B^{2q} \times 0$, and $\partial N \cap (S_i^q \cup S_i^{q+1})$ is another such pair in $B^{2q} \times 1$.

Let $f: S_1^q \cup S_2^q \rightarrow \partial N \cap (S_i^q \cup S_i^{q+1})$ take S_2^q homeomorphically onto $\partial N \cap S_i^{q+1}$, and be such that its restriction to S_1^q is a map $S_1^q \rightarrow \partial N - S_i^{q+1} \simeq S^{q-1}$ representing a generator of $\pi_q(S^{q-1})$. Note that if $q = 3$ then $\pi_q(S^{q-1}) \cong \mathbf{Z}$, and if $q > 3$ then it is $\mathbf{Z}/2\mathbf{Z}$. Since $q \geq 3$, we can homotop $f|_{S_1^q}: S_1^q \rightarrow \partial N - S_i^{q+1}$ to an embedding, using Theorem 8.1 of [H]. Extend f to an embedding $f: S_1^q \times I \rightarrow B^{2q} \times I$ so that $f|_{S_1^q \times 1}$ maps $S_1^q \times 1$ homeomorphically onto $\partial N \cap S_j^{q+1}$. Note that $B^{2q} \times I - f(S_1^q \times I) \cong (B^{2q} - S^q) \times I$ since $q \geq 3$, and so f extends to an embedding $f: (S_1^q \cup S_2^q) \times I \rightarrow B^{2q} \times I$ such that $f|_{S_2^q \times 1}$ represents a generator of $\pi_q(S^{q-1}) \cong \pi_q(B^{2q} \times 1 - f(S_1^q \times 1))$.

Now $N \cap S_i^q$ is a q -ball, and hence so is $B_i^q = \text{cl}[S_i^q - N \cap S_i^q]$. Let B_i^{2q+1} be a regular neighbourhood of B_i^q in $\text{cl}[V - N]$, meeting the boundary regularly. We can assume that $f(S_1^q \times 0) \subset B_i^{2q+1} \cap N$, since $B^{2q} \times 0 - S_i^{q+1}$ deformation retracts onto $B_i^{2q+1} \cap N$. Now we can extend $f|_{S_1^q \times I}$ to $f: S_1^{q+1} \rightarrow V$ by coning on $S_1^q \times 0$ to get a map $B^{q+1} \rightarrow B_i^{2q+1}$, and by coning on $S_1^q \times 1$ to get a map $B^{q+1} \rightarrow S_j^{q+1} \cap \text{cl}[V - N]$. Clearly $f: S_1^{q+1} \rightarrow V$ is an embedding which represents ν'_j . Similarly we can extend $f|_{S_2^q \times 1}$ to obtain an embedding representing ν'_i , and $f: S_1^{q+1} \cup S_2^{q+1} \rightarrow V$ is an embedding.

Finally we use the Whitney trick to ensure that S_i^q meets $f(S_2^{q+1})$ transversely in just one point, and similarly for S_j^q and $f(S_1^{q+1})$. Q.E.D.

PROPOSITION 2.3. *Let $q \geq 3$, and let \mathbf{u}, \mathbf{v} be nice bases of $H_q(V)$, $H_{q+1}(V)$ respectively, with ν lying over \mathbf{v} and ν' lying over \mathbf{v} . Then*

$$\nu'_i = \nu_i + \sum_{j=1}^n \lambda_{ij} u_j \circ \zeta, \quad 1 \leq i \leq n,$$

where $\lambda_{ij} = \lambda_{ji} \pmod{2}$ for all i, j .

PROOF. By the exact sequence of [W, p. 555], it is clear that $\nu'_i = \nu_i + \sum \lambda_{ij} u_j \circ \zeta$. Using Proposition 2.2(i), we can replace ν'_i by $\nu'_i + \lambda_{ii} u_i \circ \zeta$, and hence we can assume that $\lambda_{ii} = 0$ for all i . Now use Proposition 2.2(ii) to modify $\nu'_1, \nu'_2, \dots, \nu'_{n-1}$ until

$$\begin{aligned} \nu'_1 &= \nu_1, \\ \nu'_2 &= \mu_{21} u_1 \circ \zeta + \nu_2, \\ &\vdots \\ \nu'_n &= \mu_{n1} u_1 \circ \zeta + \dots + \mu_{nn-1} u_{n-1} \circ \zeta + \nu_n \end{aligned}$$

where

$$\begin{aligned} \mu_{21} &= \lambda_{21} + \lambda_{12} \pmod{2}, \\ &\vdots \\ \mu_{nj} &= \lambda_{nj} + \lambda_{jn} \pmod{2}. \end{aligned}$$

Let $V_1 = \text{cl}[V - N(\nu_1)]$, that is, the closed complement in V of a regular neighbourhood of the embedded $(q+1)$ -sphere representing ν_1 . Then $V = V_1 \cup h^q \cup h^{2q+1}$, and $H_q(V) = H_q(V_1) \oplus \langle u_1 \rangle$, $\pi_{q+1}(V) = \pi_{q+1}(V_1) \oplus \langle u_1 \circ \zeta \rangle$. Since $\nu'_2 \in \pi_{q+1}(V_1)$, we see that $\mu_{21} = 0$, and hence $\lambda_{12} = \lambda_{21} \pmod{2}$. A similar argument shows that each $\mu_{ij} = 0$ and hence $\lambda_{ij} = \lambda_{ji} \pmod{2}$. Q.E.D.

3. Seifert matrices. Let V be a $(q-1)$ -connected Seifert surface of the simple $2q$ -knot k , $q \geq 3$, with $H_q(V)$ \mathbf{Z} -torsion-free.

Let u_1, \dots, u_n be a basis of the group $H_q(V)$, and v_1, \dots, v_n the dual basis of $H_{q+1}(V)$ under the Poincaré duality pairing; thus

$$(v_i, u_j) = \delta_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n.$$

Let i_{\pm} denote the map on the homology induced by pushing a cycle off V in the \pm ve direction, and define

$$\begin{aligned} L(i_+(v_i), u_j) &= a_{ij}, \\ L(i_-(v_i), u_j) &= b_{ij}, \end{aligned} \quad 1 \leq i \leq n, \quad 1 \leq j \leq n,$$

where $L: H_{q+1}(S^{2q+2} - V) \times H_q(V) \rightarrow \mathbf{Z}$ is the linking pairing.

Note that

$$\begin{aligned} \delta_{ij} &= (v_i, u_j) = L(i_+(v_i) - i_-(v_i), u_j) \\ &= L(i_+(v_i), u_j) - L(i_-(v_i), u_j) = a_{ij} - b_{ij}. \end{aligned}$$

So $A - B = I$.

If we make a change of basis $u'_j = p_{jk}u_k$, $v'_i = q_{il}v_l$, then it is easily checked that $P' = Q^{-1}$ and the new matrices are $A_1 = QAQ^{-1}$, $B_1 = QBQ^{-1}$. Of course, P and Q are unimodular $n \times n$ matrices over \mathbf{Z} .

Now assume that \mathbf{u} and \mathbf{v} are nice dual bases of $H_q(V)$ and $H_{q+1}(V)$. Let ν_1, \dots, ν_n be elements of $\pi_{q+1}(V)$ lying over v_1, \dots, v_n .

Define matrices C , D over $\mathbf{Z}/2\mathbf{Z}$ by

$$\mathcal{L}(i_+(\nu_i), \nu_j) = c_{ij}, \quad \mathcal{L}(i_-(\nu_i), \nu_j) = d_{ij},$$

where

$$\mathcal{L}: \pi_{q+1}(S^{2q+2} - V) \times \pi_{q+1}(V) \rightarrow \mathbf{Z}/2\mathbf{Z}$$

is the homotopy linking pairing. Clearly

$$\mathcal{L}(i_+(\nu_i), \nu_j) = \mathcal{L}(\nu_i, i_-(\nu_j)) = \mathcal{L}(i_-(\nu_j), \nu_i).$$

Hence we have $c_{ij} = d_{ji}$.

For $i \neq j$, $i_+(\nu_j)$ is homotopic to $i_-(\nu_j)$ in the complement of ν_i , and so

$$\mathcal{L}(i_-(\nu_j), \nu_i) = \mathcal{L}(i_+(\nu_j), \nu_i).$$

Thus we have $d_{ji} = c_{ji}$ for $i \neq j$. Therefore C is symmetric, and $D = C$.

Suppose that ν'_1, \dots, ν'_n is another set lying over v_1, \dots, v_n ; so that $\nu'_i = \nu_i + \lambda_{ij}u_j \circ \zeta$, where $\lambda_{ij} = \lambda_{ji}$. Then

$$\begin{aligned} \mathcal{L}(i_+(\nu'_i), \nu'_j) &= \mathcal{L}(i_+(\nu_i), \nu_j) + \mathcal{L}(i_+(\lambda_{ik}u_k \circ \zeta), \nu_j) + \mathcal{L}(i_+(\nu_i), \lambda_{jl}u_l \circ \zeta) \\ &= c_{ij} + \lambda_{ik}L(i_+(u_k), \nu_j) + \lambda_{jl}L(i_+(\nu_i), u_l) \\ &= c_{ij} + \lambda_{ik}L(u_k, i_-(\nu_j)) + \lambda_{jl}L(i_+(\nu_i), u_l) \\ &= c_{ij} + \lambda_{ik}b_{jk} + \lambda_{jl}a_{il}. \end{aligned}$$

Thus $C_1 = C + \Lambda B' + \Lambda A'$.

Any unimodular integer matrix Q can be written as a product of elementary integer matrices: this is proved using the Euclidean algorithm column by column. What happens if we make such a change of basis in $H_q(V)$? Say $u'_1 = u_1 + u_2$, $u'_i = u_i$ for $2 \leq i \leq n$. Then $v'_2 = v_2 - v_1$, $v'_i = v_i$ for $i \neq 2$. We claim that ν'_1, \dots, ν'_n lies over v'_1, \dots, v'_n where $\nu'_2 = \nu_2 - \nu_1$, $\nu'_i = \nu_i$ for $i \neq 2$.

For ν'_2 can be represented by an embedded $(q+1)$ -sphere, simply by taking the connected sum of ν_1 and ν_2 with suitable orientations. A similar statement holds for u'_1 . The Whitney trick can then be used to make ν'_2 and u'_1 disjoint.

Thus if $u'_j = p_{jk}u_k$, $v'_i = q_{il}v_l$, then $\nu'_i = q_{il}\nu_l$ lies over v'_i . Hence

$$\mathcal{L}(i_+(\nu'_i), \nu'_j) = \mathcal{L}(i_+(q_{il}\nu_l), q_{jk}\nu_k) = q_{il}c_{lk}q_{jk}$$

and so $C_1 = QCQ'$.

It follows that any change of basis in $H_q(V)$, represented by a unimodular integer matrix $P = Q'^{-1}$, can be realised geometrically by embedded spheres.

PROPOSITION 3.1. *Let k, k_1 be simple \mathbf{Z} -torsion-free $2q$ -knots, $q \geq 3$, with $(q-1)$ -connected Seifert surfaces V, V_1 respectively, such that $H_q(V)$, $H_q(V_1)$ are torsion-free. Let (A, B, C) , (A_1, B_1, C_1) be Seifert matrices arising from V, V_1 . Then k is ambient isotopic to k_1 if and only if (A, B, C) is related to (A_1, B_1, C_1) as above.*

PROOF. The “only if” part has already been established. So assume that the two sets of matrices are related as above. Then by a change of basis in $H_q(V_1)$ and

$\pi_{q+1}(V_1)$ we can assume that $A_1 = A$, $B_1 = B$, $C_1 = C$. The Seifert linking forms are therefore isomorphic, and Proposition 1.1 completes the proof. Q.E.D.

We conclude this section with a realisation result.

PROPOSITION 3.2. *Let A, B be $n \times n$ integer matrices satisfying $A - B = I$, and C a symmetric $n \times n$ matrix over $\mathbf{Z}/2\mathbf{Z}$. Then for $q \geq 3$, there exists a simple \mathbf{Z} -torsion-free $2q$ -knot k with A, B, C as Seifert matrices.*

PROOF. It is implicit in the proof of Theorem II.2 of [Ke] that there is a simple \mathbf{Z} -torsion-free $2q$ -knot k_0 realising the matrices A, B by means of a $(q-1)$ -connected Seifert surface V , with $H_q(V)$ \mathbf{Z} -torsion-free. Let C_0 be the $\mathbf{Z}/2\mathbf{Z}$ matrix associated with the handle decomposition of this matrix, that is, with the basis $\mathbf{u}, \mathbf{v}, \mathbf{v}$. By altering the twisting and homotopy linking of the $(q+1)$ -handles of V , we can change C_0 to C without altering A or B . With this new embedding of V , we obtain the desired knot $k = \partial V$. Q.E.D.

4. Ambient surgery. Let k be a simple, \mathbf{Z} -torsion-free $2q$ -knot, $q \geq 3$. Assume that V_0, V_1 are two Seifert surfaces of k which are each $(q-1)$ -connected and have no \mathbf{Z} -torsion in homology. By results of Levine [L1] or Farber [F2], there exists a submanifold $W \subset S^{2q+2} \times I$ such that if $W_t = W \cap (S^{2q+2} \times t)$, then $W_0 = V_0$, $W_1 = V_1$, and $\partial W = V_0 \cup (S^{2q} \times I) \cup V_1$. Moreover we can by ambient surgery ensure that W is $(q-1)$ -connected and that $\pi_q(W)$ is \mathbf{Z} -torsion-free.

Take a handle decomposition of (W, W_0) with handles of index $q, q+1, q+2$; this is possible because (W, W_0) is $(q-1)$ -connected and so is (W, W_1) , and $\dim W = 2q+2 \geq 8$. Let $W_0 \times I$ be the collar of W_0 to which the handles are added; then $H_q(W_0 \times I \cup q\text{-handles})$ is free abelian. In these dimensions the standard handle moves can all be realised geometrically, and so because $H_q(W)$ is \mathbf{Z} -torsion-free we can arrange that for each $(q+1)$ -handle, either its attaching sphere is part of a basis for a direct summand of $H_q(W_0 \times I \cup q\text{-handles})$ or is 0 in $H_q(W_0 \times I)$. In the latter case the core of the $(q+1)$ -handle yields a basis element for $H_{q+1}(W)$.

Let N be a regular neighbourhood of $(S^{2q+2} \times 0) \cup W$, and take a handle decomposition of $S^{2q+2} \times I$ based on N . Now $S^{2q+2} \times I - W$ is $(q-1)$ -connected, and hence so is the pair $(S^{2q+2} \times I, W)$. Thus we can arrange that the handle decomposition involves only handles of index $q, q+1, q+2$, and $q+3$.

Each handle of $(W, W_0 \times I)$ yields a handle of $(N, S^{2q+2} \times I')$, where $S^{2q+2} \times I'$ is a collar neighbourhood of $S^{2q+2} \times 0$. Denote the r -handles of N by h^r , and of $(S^{2q+2} \times I, N)$ by H^r . We have say $h_1^q, \dots, h_m^q, H_1^q, \dots, H_n^q$. These must be cancelled (as a set) by the $(q+1)$ -handles, and so if we add trivial $(q+1, q+2)$ -pairs of handles, we can move the new $(q+1)$ -handles (all H^{q+1} 's) over the existing $(q+1)$ -handles to obtain a set $H_1^{q+1}, \dots, H_r^{q+1}$ which cancels the q -handles. Of course, $r \geq m+n$. From the attaching spheres we obtain an $r \times (m+n)$ matrix of integers U , with one column for each q -handle and one row for each H_i^{q+1}

$$U = \left(\begin{array}{c|c} \underbrace{\quad}_{h^q} & \underbrace{\quad}_{H^q} \end{array} \right) \}_{H^{q+1}}.$$

Moving one $(q+1)$ -handle over another adds one row to another, and similarly for q -handles and columns. Of course, we cannot move an h^q over an H^q , but otherwise there are no restrictions. Because all the q -handles are cancelled, the hcf

of the last column of U is 1. Repeated application of the Euclidean algorithm to the final column enables us, by moving the H_i^{q+1} over each other, to put U in the form $\begin{pmatrix} R & 0 \\ * & 1 \end{pmatrix}$ where R is an $(r-1) \times (m+n-1)$ matrix. Repeatedly moving H_n^q around puts U in for the form $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$. Now we can cancel H_n^q and H_r^{q+1} , replacing U by the smaller matrix R .

Continue in this way until all the H_i^q have been cancelled. Repeat the performance to obtain a matrix

$$\left\{ \overbrace{\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}}^{h^q} \right\}_{H^{q+1}}.$$

Now we can “cancel” h_m^q and H_{r-n}^{q+1} ; what this gives us is a critical level decomposition involving a q -handle added to a neighbourhood of W_0 , in other words an ambient surgery of index q performed on $V_0 = W_0$.

Repeat until all the h_i^q have been “cancelled” in this way.

Using the dual handle decomposition of (W, W) , we can repeat the procedure above. Once this has been done, we are left with a manifold (which by abuse of notation we still call W) having handles only of index $q+1$. Moreover, if N is a regular neighbourhood of $(S^{2q+2} \times 0) \cup W$, then there is a handle decomposition of $S^{2q+2} \times I$ based on N which has handles only of index $q+1$, $q+2$.

The attaching spheres of the $(q+2)$ -handles yield a square integer matrix U as above

$$U = \left(\begin{array}{c|c} \overbrace{\hspace{1cm}}^{h^{q+1}} & \overbrace{\hspace{1cm}}^{H^{q+1}} \end{array} \right) \}_{H^{q+2}}.$$

Of course, U is unimodular, and so may be written as a product of elementary matrices. The argument above goes through to show that W is the trace of ambient surgeries of index $q+1$.

5. The effect of ambient surgery. In this section we consider the effect of ambient surgery, of index q , $q+1$, and $q+2$, on a $(q-1)$ -connected Seifert surface V of the \mathbf{Z} -torsion-free simple $2q$ -knot k , where $q \geq 3$ and $H_q(V)$ is torsion-free.

(i) **INDEX q .** Since V is $(q-1)$ -connected, the attaching sphere is null-homotopic, and so we have a new basis element u_{n+1} of $H_q(U)$, where U is the new Seifert surface. Thus $H_q(U) = H_q(V) \oplus \langle u_{n+1} \rangle$. The belt sphere of the surgery supplies a new v_{n+1} , so that v_1, \dots, v_{n+1} is dual to u_1, \dots, u_{n+1} . Indeed, the belt sphere supplies $\nu_{n+1} \in \pi_{q+1}(U)$ which lies over v_{n+1} . Depending on which side of V the surgery is performed, A is replaced by

$$\begin{array}{cc} \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} A & \alpha \\ 0 & 0 \end{pmatrix} \\ \text{+ve side} & \text{-ve side} \end{array}$$

where α is a column vector of integers.

The matrix B is therefore replaced by

$$\begin{array}{cc} \begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} B & \alpha \\ 0 & -1 \end{pmatrix} \\ \text{+ve side} & \text{-ve side} \end{array}.$$

The matrix C is replaced by $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ in either case.

(ii) INDEX $q+1$. There are two possibilities, since $H_q(U)$ must be \mathbf{Z} -torsion-free. Either the attaching sphere of the surgery represents 0 in $H_q(V)$, or else it represents a primitive element, i.e., one that forms part of a basis of $H_q(V)$. These two possibilities are in fact dual to each other, for in the former case $H_r(U) = H_r(V) \oplus \mathbf{Z}$ for $r = q, q+1$, and in the latter case $H_r(V) = H_r(U) \oplus \mathbf{Z}$ for $r = q, q+1$.

Thus we need only investigate the former case, for the latter will induce the inverse effect on A , B , and C . Let u_{n+1} be the basis element of $H_q(U)$ represented by the belt sphere of the surgery, and ν_{n+1} be the element of $\pi_{q+1}(U)$ represented by $B^{q+1} \times * \subset B^{q+1} \times \partial B^{q+1}$ together with a null-homotopy of $\partial B^{q+1} \times *$. We can assume that this null-homotopy misses ν_1, \dots, ν_n , for $V - \bigcup_{i=1}^n \nu_i$ is $(q-1)$ -connected (the fibre of the sphere bundle over ν_i associated with the normal bundle is null-homotopic, using u_i) and

$$\pi_q \left(V - \bigcup_{i=1}^n \nu_i \right) \cong H_q \left(V - \bigcup_{i=1}^n \nu_i \right) \cong H_q(V).$$

Thus we can choose a null-homotopy in $V - \bigcup_{i=1}^n \nu_i$. Since $q \geq 3$ and $U - \bigcup_{i=1}^n \nu_i$ $(q-1)$ -connected, ν_{n+1} may be homotoped to an embedding, and indeed we now have ν_1, \dots, ν_{n+1} lying over v_1, \dots, v_{n+1} , a basis for $H_{q+1}(U)$.

The effect on A is

$$\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix} \quad \begin{pmatrix} A & 0 \\ \alpha & 1 \end{pmatrix}$$

+ve side -ve side

where α is a row vector of integers.

The matrix B is therefore replaced by

$$\begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix} \quad \begin{pmatrix} B & 0 \\ \alpha & 0 \end{pmatrix}.$$

+ve side -ve side

The matrix C is replaced by $\begin{pmatrix} C & \beta' \\ \beta & \gamma \end{pmatrix}$ where β is a row vector with entries in $\mathbf{Z}/2\mathbf{Z}$, and $\gamma \in \mathbf{Z}/2\mathbf{Z}$.

(iii) INDEX $q+2$. This is the inverse of a surgery of index q , and so the effect on the matrices is the inverse of (i).

REMARK. In (ii) we could choose γ to be 0, or 1. For replace the element ν_{n+1} by $\nu'_{n+1} = \nu_{n+1} + u_{n+1} \circ \varsigma$. This can be represented by an embedded sphere, and the effect on C is to replace γ by $\gamma + 1 \pmod{2}$.

PROPOSITION 5.1. *For $q \geq 3$, each of the above algebraic operations on A , B , C , may be realised geometrically by an ambient surgery on V .*

PROOF. (i) INDEX q . Suppose we have to realise the matrices

$$\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} B & \alpha \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Fatten V to a tubular neighbourhood $V \times I$, and embed a sphere S^q in $S^{2q+2} - V \times I$ so that its linking number with v_i is α_i ($1 \leq i \leq n$). Because the codimension is at least 3, S^q is unknotted in S^{2q+2} and so has trivial normal bundle $S^q \times B^{q+2}$.

Take the boundary connected sum of $V \times I$ and $S^q \times B^{q+2}$, using the positive side of $V \times I$. This is the trace of the ambient surgery.

(ii) INDEX $q + 1$. Suppose that we have to realise the matrices

$$\begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ \alpha & -1 \end{pmatrix}, \quad \begin{pmatrix} C & \beta' \\ \beta & \gamma \end{pmatrix}.$$

Once again, fatten V to $V \times I$, and embed a sphere S^{q+1} in $S^{2q+2} - V \times I$ so that it has linking number α_i with u_i ($1 \leq i \leq n$), and homotopy linking β_i with ν_i ($1 \leq i \leq n$). Note that there is no difficulty about obtaining an embedding: we just use Theorem 8.1 of [H]. Take a regular neighbourhood $S^{q+1} \times B^{q+1}$ of S^{q+1} , and then take the boundary connected sum of this with $V \times I$, attaching it on the positive side. The new basis elements are realised by $S^{q+1} \times \text{point}$ and $\text{point} \times S^q$, both contained in $S^{q+1} \times \partial B^{q+1}$, and clearly we realise the desired matrices, except possibly for γ . But by the remark above, if we can realise $\gamma + 1$, then we can realise γ .

Of course, similar arguments hold in the case of the negative side of V . To realise the dual case of index $q + 1$, we realise the reduced matrices by a new knot and Seifert surface V_1 , using Proposition 3.2, perform ambient surgery on V_1 to realise the original matrices, and then appeal to Proposition 3.1 to show that the new Seifert surface is ambient isotopic to V . Hence we have an ambient surgery which realises the algebraic move. A similar argument applies to index $q + 2$, which is dual to index q .

6. Presentation of the F -form. Recall that u_1, \dots, u_n is a basis of $H_q(V)$ and v_1, \dots, v_n the dual basis of $H_{q+1}(V)$. Let ν_1, \dots, ν_n be elements of $\pi_{q+1}(V)$ lying over v_1, \dots, v_n . Then $S^{2q+2} - V$ is $(q - 1)$ -connected, and we can choose bases $\alpha_1, \dots, \alpha_n$ of $H_q(S^{2q+2} - V)$ and β_1, \dots, β_n of $H_{q+1}(S^{2q+2} - V)$ as follows. The bases u_1, \dots, u_n and ν_1, \dots, ν_n are represented by spheres S_1^q, \dots, S_n^q and $S_1^{q+1}, \dots, S_n^{q+1}$ embedded in V , such that S_i^q meets S_j^{q+1} transversely in δ_{ij} points, and otherwise these spheres are disjoint. Each sphere is unknotted in S^{2q+2} , having codimension at least $q + 1 > 3$, and so has trivial normal bundle. Let α_i be represented by the boundary of a fibre of the bundle over S_i^{q+1} , and β_i by the boundary of a fibre of the bundle over S_i^q . Let $\gamma_i \in \pi_{q+1}(S^{2q+2} - V)$ lie over β_i , and note that these elements can be chosen so that

$$L(\beta_i, u_j) = \delta_{ij} = L(v_i, \alpha_j),$$

$$1 \leq i, j \leq n.$$

$$\mathcal{L}(\gamma_i, \nu_j) = 0,$$

Let $i_+(v_i) = h_{ij}\beta_j$; then

$$a_{ij} = L(i_+(v_i), u_j) = L(h_{ik}\beta_k, u_j) = h_{ik}L(\beta_k, u_j) = h_{ik}\delta_{kj} = h_{ij}.$$

Thus $i_+(v_i) = a_{ij}\beta_j$, and similarly $i_-(v_i) = b_{ij}\beta_j$.

We can write $i_+(\nu_i) = a_{ij}\gamma_j + e_{ij}\alpha_j \circ \varsigma$, and using the fact that $(\alpha_k \circ \varsigma, \nu_j) \equiv L(\alpha_k, v_j) \equiv \delta_{kj} \pmod{2}$, we see that

$$\begin{aligned} c_{ij} &= \mathcal{L}(i_+(\nu_i), \nu_j) = \mathcal{L}(a_{ik}\gamma_k + e_{ik}\alpha_k \circ \varsigma, \nu_j) \\ &= a_{ik}\mathcal{L}(\gamma_k, \nu_j) + e_{ik}\mathcal{L}(\alpha_k \circ \varsigma, \nu_j) = e_{ik}\delta_{kj} = e_{ij}. \end{aligned}$$

Thus $i_+(\nu_i) = a_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$, and similarly $i_-(\nu_i) = b_{ij}\gamma_j + c_{ij}\alpha_j \circ \varsigma$.

By standard arguments, $tA - B$ is a presentation matrix for $H_{q+1}(\tilde{K})$ as a Λ -module; that is,

$$H_{q+1}(\tilde{K}) \cong (\beta_1, \dots, \beta_n: (ta_{ij} - b_{ij})\beta_j, 1 \leq i \leq n).$$

Let $i_-(u_i) = f_{ij}\alpha_j$; then

$$\begin{aligned} a_{ij} &= L(i_+(v_i), u_j) = L(v_i, i_-(u_j)) = L(v_i, f_{jk}\alpha_k) \\ &= f_{jk}L(v_i, \alpha_k) = f_{jk}\delta_{ik} = f_{ji} \end{aligned}$$

so that $i_-(u_i) = a_{ji}\alpha_j$. And similarly $i_+(u_i) = b_{ji}\alpha_j$. Thus $H_q(\tilde{K})$ is presented as a Λ -module by $tB' - A'$. Allowing α_i, β_j to represent their images in $H_q(\tilde{K})$, $H_{q+1}(\tilde{K})$, respectively, the Blanchfield pairing is given (up to sign) by the formula

$$\langle \beta_i, \alpha_j \rangle \equiv (t-1)(tA - B)_{ij}^{-1} \pmod{\Lambda}.$$

There is a map of Γ -modules

$$\begin{aligned} &(\alpha_1 \circ \varsigma, \dots, \alpha_n \circ \varsigma, \gamma_1, \dots, \gamma_n: \\ &(ti_+(u_i) - i_-(u_i)) \circ \varsigma, ti_+(\nu_i) - i_-(\nu_i), 1 \leq i \leq n) \rightarrow \Pi_{q+1}(\tilde{K}), \end{aligned}$$

that is,

$$\begin{aligned} &(\alpha_1 \circ \varsigma, \dots, \alpha_n \circ \varsigma, \gamma_1, \dots, \gamma_n: \\ &(tb_{ji} - a_{ji})\alpha_j \circ \varsigma, (ta_{ij} - b_{ij})\gamma_j + (t-1)c_{ij}\alpha_j \circ \varsigma) \rightarrow \Pi_{q+1}(\tilde{K}). \end{aligned}$$

Denoting this presentation Γ -module by N , and the Γ -module

$$(\alpha_1 \circ \varsigma, \dots, \alpha_n \circ \varsigma: (tb_{ji} - a_{ji})\alpha_j \circ \varsigma, 1 \leq i \leq n)$$

by M , and the Γ -module

$$(\beta_1, \dots, \beta_n: (ta_{ij} - b_{ij})\beta_j, 1 \leq i \leq n)$$

by P , we see that there is a commutative diagram

$$\begin{array}{ccccc} M & \rightarrow & N & \rightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}_q(\tilde{K}) & \rightarrow & \Pi_{q+1}(\tilde{K}) & \rightarrow & \mathcal{H}_{q+1}(\tilde{K}) \end{array}$$

of Γ -modules, both rows being short exact sequences.

The first and third vertical arrows are isomorphisms, and so by the five-lemma is the middle one. Hence we have a presentation for $\Pi_{q+1}(\tilde{K})$ as a Γ -module. As in [K] the hermitian pairing is given by

$$[\gamma_i \gamma_j] \equiv (t-1)[(tA - B)^{-1}(t^{-1}C' - C)(B' - t^{-1}A')]_{ij}.$$

7. Seifert matrices and F -forms. Let A, B, C be the Seifert matrices of a simple \mathbf{Z} -torsion-free $2q$ -knot k , $q \geq 3$, arising from a choice of basis \mathbf{u} of $H_q(V)$ where V is a $(q-1)$ -connected Seifert surface of k , with $H_q(V)$ torsion free. Of course, we also have in mind a choice of $\nu \in \pi_{q+1}(V)$ lying over \mathbf{v} , the dual basis of \mathbf{u} . In §3 we investigated the way in which A, B, C change when \mathbf{u} and ν are changed, and in §5 the way an ambient surgery on V affects them. Call the equivalence relation generated by these changes *F-equivalence*.

THEOREM 7.1. *Let A, B be $n \times n$ integer matrices satisfying $A - B = I$, and C a symmetric $n \times n$ matrix over $\mathbf{Z}/2\mathbf{Z}$; and let A_1, B_1, C_1 be another such set of $m \times m$ matrices. Then (A, B, C) is F -equivalent to (A_1, B_1, C_1) if and only if they present isomorphic F -forms.*

PROOF. Choose an integer $q \geq 4$, and by Proposition 3.1 realise A, B, C as Seifert matrices of a simple \mathbf{Z} -torsion-free $2q$ -knot k , and A_1, B_1, C_1 as Seifert matrices of a similar knot k_1 . Then if the F -forms of k and k_1 are isomorphic, by Corollary 11.3 of [K], k and k_1 are ambient isotopic. Hence the Seifert surfaces, V and V_1 say, are related by a sequence of ambient surgeries and so (A, B, C) is F -equivalent to (A_1, B_1, C_1) . Conversely, if the matrices are F -equivalent we can realise the algebraic moves by ambient surgeries; hence k and k_1 are ambient isotopic and so the F -forms are isometric. Q.E.D.

The F -form of a simple \mathbf{Z} -torsion-free $2q$ -knot is defined in [K3] for $q \geq 4$. We are in a position to extend this to the case $q = 3$.

THEOREM 7.2. *Every simple \mathbf{Z} -torsion-free $2q$ -knot, $q \geq 3$, gives rise to an F -form which is an invariant of the knot.*

PROOF. The case $q > 3$ is dealt with in [K]. For $q = 3$, the Whitehead exact sequence

$$2H_q(\tilde{K}) \rightarrow H_q(\tilde{K}) \rightarrow \pi_{q+1}(\tilde{K}) \rightarrow H_{q+1}(\tilde{K})$$

holds, and so we have the necessary modules and sequences. (Recall that $\tilde{K} \rightarrow K$ is the infinite-cyclic cover of the exterior K of the knot k .)

In [K, §1], the pairing $\{ , \}$ is well defined except for the choice of α . If we take $\alpha = \Delta\bar{\Delta}$ as in [K, Corollary 1.3], then this is a canonical choice, being a knot invariant, and so we obtain a homotopy linking which is well defined for $q = 3$.

Alternatively, we could define the F -form by means of the presentation in terms of A, B, C , and use Theorem 7.1. Q.E.D.

An axiomatic description of F -forms is given in [K3], together with a realisation theorem for $q \geq 4$. Again, we extend that result to the case $q = 3$.

THEOREM 7.3. *Every F -form can be realised by a simple \mathbf{Z} -torsion-free $2q$ -knot, $q \geq 3$.*

PROOF. For $q > 3$, the F -form can be realised by a simple \mathbf{Z} -torsion-free $2q$ -knot. From this we obtain Seifert matrices A, B, C which present the F -form. And Proposition 3.2 yields a simple \mathbf{Z} -torsion-free 6-knot with these as Seifert matrices, hence with F -form isometric to the given one. Q.E.D.

Finally we show that F -form is a complete invariant of the knot when $q \geq 3$.

THEOREM 7.4. *Two simple \mathbf{Z} -torsion-free $2q$ -knots, $q \geq 3$, with isometric F -forms, are ambient isotopic.*

PROOF. By [K, Theorem 11.1] the result is true for $q \geq 4$. For $q = 3$, let $(A, B, C), (A_1, B_1, C_1)$ be Seifert matrices arising from Seifert surfaces V, V_1 of the knots k, k_1 . Since the F -forms are isometric, the Seifert matrices are F -equivalent by Theorem 7.1. The algebraic moves on the matrices can be realised geometrically, by Proposition 5.1, and so k and k_1 are isotopic by Proposition 3.1. Q.E.D.

REFERENCES

- [F1] M. Sh. Farber, *Isotopy types of knots of codimension two*, Trans. Amer. Math. Soc. **261** (1980), 185–209.
- [F2] —, *The classification of simple knots*, Russian Math. Surveys **38** (1983), 63–117.
- [H] J. F. P. Hudson, *Piecewise linear topology*, Benjamin, New York, 1969.
- [K1] C. Kearton, *Classification of simple knots by Blanchfield duality*, Bull. Amer. Math. Soc. **79** (1973), 952–955.
- [K2] —, *An algebraic classification of some even-dimensional simple knots*, Topology **15** (1976), 363–373.
- [K3] —, *An algebraic classification of certain simple even-dimensional knots*, Trans. Amer. Math. Soc. **276** (1983), 1–53.
- [Ke] M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.
- [Ko1] S. Kojima, *A classification of some even-dimensional fibred knots*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 671–683.
- [Ko2] —, *Classification of simple knots by Levine pairings*, Comment. Math. Helv. **54** (1979), 356–367.
- [L1] J. Levine, *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16.
- [L2] —, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970), 185–198.
- [T1] H. F. Trotter, *On S-equivalence of Seifert matrices*, Invent. Math. **20** (1973), 173–207.
- [T2] —, *Knot modules and Seifert matrices*, Lecture Notes in Math., vol. 685, Springer-Verlag, Berlin and New York, 1978, pp. 291–299.
- [W] G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, Berlin and New York, 1978.

SCHOOL OF MATHEMATICS, PHYSICS, COMPUTING AND ELECTRONICS, MACQUARIE UNIVERSITY, N.S.W. 2109 AUSTRALIA

DEPARTMENT OF MATHEMATICS, SCIENCE LABORATORIES, DURHAM UNIVERSITY, DURHAM DH1 3LE, ENGLAND